B.Sc. Part I (Hons) Ist Paper

(TRIGONOMETRY)

HYPERBOLIC FUNCTIONS

Sums. Q. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$ then prove that

(a) $e^{2\phi} = \pm \cot\frac{\alpha}{2}$ (b) $2\theta = n\pi + \frac{\pi}{2} + \alpha$.

Soln. Given that

$$\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha \quad \text{--- (1)}$$

Replacing i by $-i$, in eq(1), we get

$$\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha \quad \text{--- (2)}.$$

Now, $2i\phi = (\theta + i\phi) - (\theta - i\phi)$

$$\Rightarrow \tan 2i\phi = \tan[\underbrace{(\theta + i\phi)}_{A} - \underbrace{(\theta - i\phi)}_{B}]$$

$$\Rightarrow \tan 2i\phi = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi).\tan(\theta - i\phi)}$$

$$\Rightarrow \tan 2i\phi = \frac{\tan\alpha + i\sec\alpha - \tan\alpha + i\sec\alpha}{1 + (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$\Rightarrow \tan 2i\phi = \frac{2i\sec\alpha}{1+(\tan^2\alpha - i^2\sec^2\alpha)} = \frac{2i\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}$$

$$\Rightarrow \tan 2i\phi = \frac{2i\sec\alpha}{\sec^2\alpha+\sec^2\alpha} = \frac{2i\sec\alpha}{2\sec^2\alpha}$$

$$\Rightarrow \tan 2i\phi = \frac{i}{\sec\alpha}$$

$$\boxed{\because \tan i\theta = i\tanh\theta}$$

$$\Rightarrow i\tanh 2\phi = \frac{i}{\sec\alpha} \Rightarrow \tanh 2\phi = \cos\alpha$$

$$\boxed{\because \tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}}$$

$$\Rightarrow \frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \frac{\cos\alpha}{1}$$

By componendo and dividendo,

$$\frac{e^{2\phi}+e^{-2\phi}+e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}-e^{2\phi}+e^{-2\phi}} = \frac{1+\cos\alpha}{1-\cos\alpha}$$

$$\Rightarrow \frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{2\cos^2\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}$$

$$\Rightarrow e^{4\phi} = \cot^2\frac{\alpha}{2}$$

Taking square root, we have

$$e^{2\phi} = \pm\cot\frac{\alpha}{2}.$$ Part(a) Proved

Again,

$$2\theta = (\theta+i\phi)+(\theta-i\phi)$$

$$\Rightarrow \tan 2\theta = \tan[(\theta+i\phi)+(\theta-i\phi)]$$

$$\Rightarrow \tan 2\theta = \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi).\tan(\theta-i\phi)}$$

$$\Rightarrow \tan 2\theta = \frac{\tan\alpha + i\cancel{\sec\alpha} + \tan\alpha - i\cancel{\sec\alpha}}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}$$

$$= \tan 2\theta = \frac{2\tan\alpha}{1-(\tan^2\alpha - i^2\sec^2\alpha)} = \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)}$$

$$\Rightarrow \tan 2\theta = \frac{2\tan\alpha}{1-\tan^2\alpha-\sec^2\alpha}.$$

$$\Rightarrow \tan 2\theta = \frac{2\tan\alpha}{-\tan^2\alpha-(\sec^2\alpha-1)} = \frac{2\tan\alpha}{-\tan^2\alpha-\tan^2\alpha}$$

$$\Rightarrow \tan 2\theta = \frac{2\tan\alpha}{-2\tan^2\alpha} = -\frac{1}{\tan\alpha}$$

$$\Rightarrow \tan 2\theta = -\cot\alpha$$

$$\Rightarrow \tan 2\theta = \tan\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow 2\theta = n\pi + \left(\frac{\pi}{2}+\alpha\right)$$

Part (b) Proved